LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS

MTH 201 - CALCULUS 3

FINAL EXAM – FALL 2015

Duration: 120 minutes

Name: **ID#:**

Instructor's name: Solutions

NO CALCULATORS ALLOWED

- **This exam consists of 10 pages and 9 problems.**
- **Answer the questions below on the space provided. You can use the back pages for scratch or for more space for your answers. Please specify.**
- **Make sure you justify all your answers.**

Question Number	Grade
1. 10%	
2. 18%	
3. 12%	
4. 16%	
5. 8%	
6. 8%	
7. 8%	
8. 10%	
9. 10%	
TOTAL	

(10%) **Problem 1:** Evaluate the following improper integral

$$\int_1^2 \frac{du}{u^2 - 2u}$$

$$\int_1^2 \frac{du}{u(u-2)} = \lim_{t \to 2^-} \int_1^t \frac{du}{u(u-2)}$$

$$\frac{A}{u} + \frac{B}{u-2} = \frac{1}{u(u-2)} \Rightarrow A = -1/2 \qquad B = \frac{1}{2}.$$

$$\therefore = \lim_{t \to 2^-} \int_1^t \left(\frac{1/2}{u-2} - \frac{1/2}{u} \right) du$$

$$= \lim 1/2 \ln|u-2| - \frac{1}{2} \ln|u| \Big|_1^t$$

$$= \lim_{t \to 2^-} \ln \sqrt{\left| \frac{u-2}{u} \right|} \Big|_1^t \longrightarrow -\infty \quad \therefore \text{ diverges}$$

(18%) **Problem 2:** Determine whether the following series converge or diverge.

1. $\sum_{n=2}^{\infty} \frac{(\ln n)^{\frac{1}{2}}}{n^2}$ conv.

$$(\ln n)^{1/2} < n^{0.1} \Rightarrow$$

$$\frac{(\ln n)^{1/2}}{n^2} < \frac{1}{n^{1.9}}$$

But since $\Sigma \frac{1}{n^{1.9}}$ conv

p-series $p > 1 \Rightarrow$ our int. conv. by DCT

2. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges since

$$a_n = \frac{n^n}{n!} \longrightarrow \infty \quad \left(\frac{\cancel{fast}}{slow}\right)$$

$\therefore a_n \not\to 0 \Rightarrow \Sigma a_n$ div. by n^{th} term test.

3. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n+2}{3n-1}\right)^n \approx \Sigma (-1)^{n+1}\left(\frac{1}{3}\right)^n = -\Sigma\left(\frac{-1}{3}\right)^n$

geom. series with $r = -1/3$

$\Rightarrow |r| < 1 \Rightarrow$ geom. series conv.

(12%) **Problem 3:** Using Maclaurin series, approximate the following integral

$$\int_0^1 \frac{1-\cos x}{x}\,dx,$$

with an error of magnitude no greater than 10^{-4}. You just need to write the inequality needed for finding the number of terms in the approximation.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$

$\frac{1-\cos x}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \cdots$

$\therefore \int_0^1 \frac{1-\cos x}{x}\,dx = \int_0^1 \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \cdots$

$= \frac{x^2}{2(2!)} - \frac{x^4}{4(4!)} + \frac{x^6}{6(6!)} - \cdots \Big|_0^1$

$= \frac{1}{2(2!)} - \frac{1}{4(4!)} + \frac{1}{6(6!)} - \cdots$

Need to stop at term $\approx \frac{1}{10^4}$

$= \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n)(2n)!}$

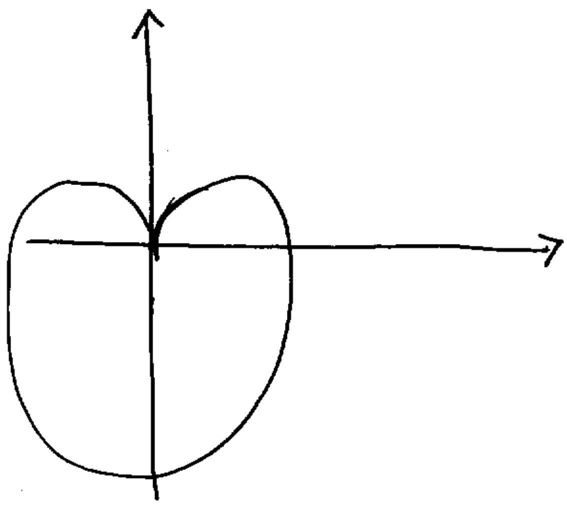
(16%) **Problem 4:** Consider the polar curve

$$r = f(\theta) = 1 - \sin\theta,$$

where $\theta \in [0, 2\pi]$.

1. Discuss its symmetries

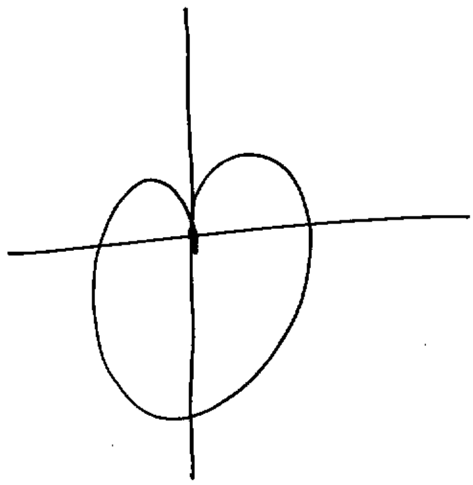
the y-axis



2. Find its slope at the origin

$r = 0 \Rightarrow \theta = \pi/2 \Rightarrow$ the y-axis is the tangent line @ origin $\Rightarrow$ slope $= \pm\infty$

3. Plot it as accurately as possible



4. Find the area lying inside the curve $r = f(\theta)$ in the first quadrant. $\int\int dA =$

$$= \int_0^{\pi/2}\int_0^{1-\sin\theta} r\,dr\,d\theta. = \int_0^{\pi/2} \frac{(1-\sin\theta)^2}{2}\,d\theta.$$

$$\cancel{= \theta + \cos\theta \Big|_0^{\pi/2} = \frac{\pi}{2} - 1}$$

(8%) **Problem 5:** Evaluate the following limit or show that it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{y\sin x}{x^2+y^2}$$

Using polar coordinates

$$= \lim \frac{r\sin\theta \sin(r\cos\theta)}{r^2} = \frac{\sin\theta \sin(r\cos\theta)}{r}$$

Or Try $y = mx$.

$$\lim_{x\to 0} \frac{mx \cdot \sin x}{x^2 + m^2x^2} = \lim \frac{m \sin x}{x(1+m^2)}$$

$$= \lim \frac{m}{1+m^2} \Rightarrow \text{Depends on } m \Rightarrow \text{No Limit.}$$

$$\frac{\sin x}{x} \to 1 \Rightarrow$$

(8%) **Problem 6:** Let

$$f(x,y) = x^3 - 4xy^2 - \sin(xy)$$

Find f_{xx}, f_{yy} and f_{xy}

$$f_x = 3x^2 - 4y^2 - y\cos xy$$

$$f_{xx} = 6x + y^2\sin(xy)$$

$$f_{xy} = -8y - \cos xy + xy\sin(xy)$$

$$f_y = -8xy - x\cos(xy)$$

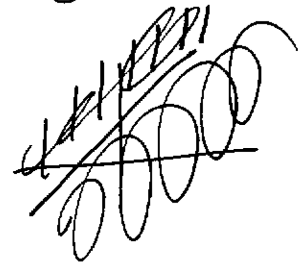
$$f_{yy} = -8x + x^2\sin(xy)$$

(8%) **Problem 7:** Let

$$f(x,y) = \sqrt{2+x-y}$$

Find the domain (sketch it in the plane), range and describe its level curves.

Domain: $2 + x - y \geq 0$ $\qquad y \leq x + 2$



Range: $z \geq 0$

Level curves:

$(S) \cap (z = c)$. $\qquad \sqrt{2+x-y} = c$

$2 + x - y = c^2$

$y = x + (c^2 + 2)$

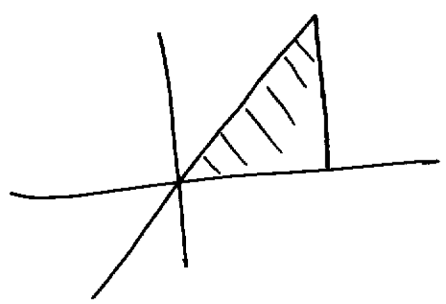
Lines // 1º bissector with y-intercept (c^2+2)

(always > 2)

(10%) **Problem 8:** Evaluate the double integral:

$$\int_0^1 \int_y^1 3x\, e^{x^3}\, dx\, dy$$

$y < x < 1 \qquad 0 < y < 1$



$$= \int_0^1 \int_0^x 3x e^{x^3}\, dy\, dx = \int_0^1 x y e^{x^3} \Big|_0^x \, dx.$$

$$= \int_0^1 x^2 e^{x^3}\, dx = \frac{1}{3} e^{x^3} \Big|_0^1$$

$$= \frac{1}{3}(e - 1)$$

(10%) **Problem 9:** Using polar coordinates, evaluate the double integral:

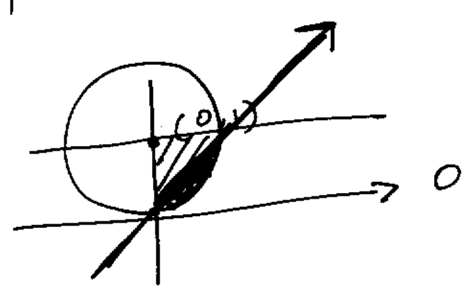
$$\int_0^1 \int_y^{\sqrt{2y-y^2}} x\,dx\,dy$$

$x = \sqrt{2y-y^2}$ $\quad x > 0$

$x^2 = 2y - y^2 \Rightarrow x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$

$\Rightarrow$



$r = 2\sin\theta.$

$$\int_0^{\pi/4} \int_{r=0}^{r=2\sin\theta} r\cos\theta \; r\,dr\,d\theta.$$

$$= \int_0^{\pi/4} \frac{r^3}{3}\cos\theta \Big|_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{8}{3}\sin^3\theta\cos\theta\,d\theta = \frac{8}{3}\frac{\sin^4\theta}{4}\Big|_0^{\pi/4}$$

$$= \frac{2}{3}\,\frac{1/4}{4} =$$

$$= \frac{1}{24}$$